A Cash Flow Based Multi-period Credit Risk Model

Tsung-kang Chen *   Hsien-hsing Liao **

First Version: May 15, 2004
Current Version: August 20, 2004

ABSTRACT

Many credit risk models have been proposed in the literature. According to their assumptions and emphases, they can be roughly grouped into two categories, “structural-form” models and “reduced-form” models. The former focuses on constructing the distribution of asset values and then estimating one-period probability of default and recovery rate; the latter relies on exogenous information such as credit rating and recovery rate which are not related to asset values. Within the frameworks of the above two categories, few studies use stochastic cash flow model to assess firm’s credit risk. Based upon the two significant cash flow characteristics—“mean-reversion” and “allowing positive or negative values” and the concept of varying coefficient model, the study develops a “Time-dependent stochastic cash flow model”. To consider future industrial economic state changes’ impacts on a firm’s cash flow, we also construct a stochastic model of industrial economic state. The information forecasted from the state model is used as the base for adjusting the parameters of the time-dependent cash flow model. To perform a multi-period firm valuation, this cash flow model needs only publicly available information of corporate finance and the industrial economic state (i.e. the industrial cyclicity information). With the information of a firm’s value and debt, a “Cash Flow Based Multi-period Credit Risk Model” can be built. We can then price the corporate debt by combining the expected recovery rate generated from our model and the concept of defaultable bond pricing of Jarrow-Turnbull Model (1995).

* Ph.D Student, Department of finance, National Taiwan University, Taiwan, r91723010@ntu.edu.tw
** Associate Professor, Department of finance, National Taiwan University, Taiwan, hliao@ntu.edu.tw
I. Introduction

Many credit risk models have been proposed in the literature. According to their assumptions and emphases, they can be roughly categorized into “structural-form” models and “reduced-form” models. The “structural-form” credit models focus on constructing the distribution of a firm’s asset value and estimating probability of default (later denoted as PD) and recovery rate (later denoted as RR). The “reduced-form” credit models on the other hand stress the “non-asset-value related information”, such as credit rating (Litterman and Iben, 1991; Jarrow, Lando and Turnbull, 1997) and recovery rate. They estimate and price a firm’s credit risk by observable market credit spreads. This makes their models closely linked to market’s current situation. Recently, some studies take into consideration systematic risk and develop the so-called “single systematic factor models”. These models investigate the relationship between macroeconomic factors and credit risk variables, such as PD and RR. They use historical data to establish a regression model to do a concurrent credit risk analysis. Although PD and RR are both externally determined in “Single Systematic Factor Model”, we can still find a negative relationship between PD and RR because they both linked to a same common systematic risk factor. Therefore the models’ characteristics are different from the above two traditionally models.
Despite the fact that “reduced-form” credit risk models better match market current reality, they rely on exogenous information such as credit rating and RR rather than on a firm’s financial information. Regarding the “structural-form” models, though they use historical financial data to do credit risk analysis, few of them can generate reasonable multi-period value distribution of the firm.\textsuperscript{6} Within the frameworks of the above two traditional credit risk models and the recently developed “single systematic factor models”, few studies use stochastic cash flow model to evaluate firm’s credit risk.\textsuperscript{7} It’s because people generally think it difficult to estimate a firm’s future cash flows, and hence no one had ever developed an applicable model to describe the stochastic characteristics of cash flow. Through our observations of cash flow, however, we discover that the behavior of cash flow exhibits some stochastic characteristics, such as mean-reversion and allowing positive or negative values. In addition, these time-varying cash flows behaviors are influenced by changes of industrial economic states.

It is widely accepted that a firm’s cash flows can, in most cases, honestly reflect its operating value. In order to obtain a firm’s value distributions, this study starts in building a stochastic cash flow model that is reasonably good in describing aforementioned cash flow characteristics. To allow the cash flow model reflecting the changes of industrial economic states, the cash flow model is designed to be time dependent. That is the parameters of the stochastic cash flow model are time varying and alter according to the changes of industrial economic states. Adopting the concept of varying coefficient model, we construct a stochastic model of industrial economic state, using industrial cyclical factors as proxies for the industrial economic states. The information forecasted by the industrial economic state model is used as the base for adjusting the parameters of the time-dependent cash flow model, which we call a \textit{“Time-dependent stochastic cash flow model”}.

\textsuperscript{6} Black-Scholes (1973) assume stock price is lognormally-distributed. However, this assumption is not supported by empirical results. So it causes default probability will be undervalued and recovery rate will be overvalued in Merton model (1974). So based on Merton model, firm’s value distribution can not be reasonably estimated. In fact, if you can find out the real proxy of firm value, the evaluation efficiency of structural-form models will be obviously improved.

\textsuperscript{7} To our best knowledge, there is no publicly distributed cash flow based credit risk model.
flow model”. With the cash flow model, we can generate a firm’s future cash flows and subsequently can obtain the firm’s value distributions in future periods. Knowing a firm’s multi-period value distributions and the firm’s debt information, we are able to build a multi-period credit risk model, which we call a “Cash flow-based multi-period credit risk model”.

PD and RR are two major indicators in measuring credit risk. Structural-form credit risk models are mainly developed within the framework of Merton’s option theory (Merton, 1974) and other mathematical methods to derive a “single-period” firm’s asset value distribution. With the value distribution and the firm’s targeting debt, PD and RR can be obtained endogenously. However, due to model constraints, they usually can’t get correct value distribution of the firm’s assets. To resolve the problem, models such as KMV (KMV, 1999) were developed by heavily relying upon empirical data as base for model calibration. On the other hand, reduced-form credit risk models simplify their credit risk measuring process under the framework of exogenous RR and non-asset-value related information. Therefore RR is exogenous and independent from PD.

However, most credit models are basically established in a “single-period” framework. Few models can be reasonably extended to multi-period. For traditional structural-form models, they cannot be extended to multi-period structure because they are all constrained by Merton’s framework. For the single systematic factor models, currently they can be only used to analyze the current credit risks. For reduced form models, although some of them can be extended to multi-period such as Jarrow and Turnbull (1995), Jarrow, Lando

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8 After simulation, these models can only generate a single-period result. To produce a multi-period outcome, several times of simulation then are needed. But this extension is economic-meaningless because the parameters used in the simulations are perhaps estimated from different periods or fixed in the future periods. For the former cause, it will lead to inconsistence for the future estimates; and for the latter cause, it will react on the past information.
9 The stock returns are assumed to follow the normal distribution in this kind of model; but actually “return distribution is skewed to the left, and has a higher peak and two heavier tails than those of normal distribution” (S. G. Kou, 2002)
10 In reduced-form models, they assume an exogenous RR that is either a constant or a stochastic variable independent from PD. And RR is usually assumed to follow a beta distribution. And the recovery rates are usually estimated from historical data or past experience of similar companies.
11 Most of reduced-form credit risk models also employ credit rating information such as prices of different credit ratings (JT, 1995), historical credit ratings’ changes data (JLT, 1997; KK, 1998), and diversity score calculation from credit rating data (portfolio models). However, credit ratings are backward information because the credit ratings can’t immediately react on the changes of a firm’s value in the future.
12 J-T model based on the no-arbitrage assumption thinks that the rising probability of interest rate in the default risk-free bond market, the default probability of defaultable bond market will be solved the unique solution when future RR and bond prices are known.
and Turnbull (1997)\textsuperscript{13}, and Duffie and Wang (2004), they are subjected to unrealistic assumptions or theoretical basis. J-T model can be extended to multi-period by using the concept of B.D.T. model (Black-Derman-Toy) but it will be limited by the assumptions of no-arbitrage or risk-neutral\textsuperscript{14} condition and by an exogenous RR. Regarding JLT model, its extension to multi-period will be not only limited by the risk-neutral condition but also by the memoryless assumption of Markov’s chain. Duffie and Wang (2004) think that PD can be decided by distance-to-default and personal income growth. They use the time-series model to predict the future estimates of distance-to-default\textsuperscript{15} and personal income growth. So they primarily focus on the distance-to-default and view it as a basis of bankruptcy prediction. But, does distance-to-default actually have the stochastic characteristic of mean-reversion\textsuperscript{16} as they assert? Further debates are unavoidable. Moreover, distance-to-default is calculated by the relationship between past asset and liability data. It may not be able to reflect a firm’s future values\textsuperscript{17}. In addition, recovery rate is also exogenous in this model.

While doing a multi-period extension, past and newly developed credit risk models have some theoretical limitations as follows: recovery rate being exogenous, the disputable rationality of models’ assumptions, firms’ real value not being appropriately reflected\textsuperscript{18}. In this study, we start from the basic concept of structural-form models and find an

\textsuperscript{13} They utilize “Markov chains” to explore the dynamic process of credit rating in single period. Due to the memoryless assumption of Markov chain, the future credit rating changes don’t be influenced by the past credit rating when applying in multi-period researches. This doesn’t conform to the real world. According to the empirical results of Carty and Lieberman (1997), they discover that the future movement of credit rating will be influenced by the past credit rating. It demolishes Markov chain’s assumption. So we will relax the memoryless assumption when applying in multi-period researches.

\textsuperscript{14} Since the purpose of most intensity models is valuation of bonds or derivatives, it makes the model developers assume risk-neutrality to simplify valuation process.

\textsuperscript{15} They collect 28,612 historical quarter data of firms’ distance-to-default (calculated by using KMV model). The calculation method is the same as Crosbie and Bohn (2002), Vassalou and Xing (2004). Therefore, distance-to-default will have the shortcoming of non-reasonable assumption of stock price.

\textsuperscript{16} When a firm’s distance-to-default (DD) suddenly enlarges, PD will be lower and its credit rating will be higher. If a firm’s DD really has the stochastic characteristic of mean-reversion, it will be decreased to the long-run average DD and then PD will increase and credit rating will downgrade in the next period. But in real world, a higher rated company will try to maintain at least the same grade in the next period; however, a lower rated company will try to improve its current poor grade in the next period. Therefore, the dynamic process of DD or credit rating exist a phenomenon of “asymmetry”.

\textsuperscript{17} When a firm experience different business life cycle, the relationship between asset and liability will differ on different life stage. In seed stage or growth stage, debt ratio will be higher to support substantial capital expenditures; in mature stage, debt ratio will be lower. But each stage has different time period, DD can’t fully reflect the stage changes in the future because it is an absolute distance without specific economic meanings. The stage changes will exactly react on a firm’s value.

\textsuperscript{18} The mainly purpose of intensity models is to evaluate the default risk of bonds and derivatives and further to price. Intensity models usually use known price data or credit rating data of default risk bonds and default risk-free bonds instead of asset-value related information. So they don’t consider a firm’s real value so that they cannot provide a complete and systematic evaluation process based on a firm’s real value.
instrument variable, free cash flow, to calculate the firm’s value. Because a normally managed firm has strong motivation to maintain its cash flows stable or stable with an upward trend and a firm’s cash flows are severely influenced by the state of industrial state, we have the idea to develop a time-dependent stochastic cash flow model upon which we are able to construct a “Cash Flow Based Multi-periods Credit Risk Model”. With the models, we can reasonably get endogenous multi-period PD and RR and then price the corporate bond by combining the expected RR generated from our models and the concept of defaultable bond pricing of Jarrow-Turnbull Model (1995). The edifice makes our model successfully go beyond the constraints of Merton’s framework that dominates the developments of most traditional structural-form models. We can also reasonably extend the structural-form models from single-period models to multi-period models by a stochastic cash flow model that is able to incorporate information of possible changes of future economic states in the future cash flows it simulates.

The rest of the paper is divided into four sections: First, we construct a time-dependent stochastic cash flow model, including a discussion on the stochastic characteristics of cash flows, the time-dependent stochastic cash flow model, and a stochastic industrial economic state model; Second, we present a “cash flow based multi-period credit risk model”; Third, we empirically examine effectiveness of our model. In the last section we conclude this study.

II. The Time-Dependent Cash Flow Model

In this section, firstly, we explore the characteristics of firm’s cash flow; and second we discuss stochastic models that can appropriately describe cash flow characteristics. Third, we construct our cash flow models based upon previous discussion. Finally, to consider future industrial economic state changes’ impacts on a firm’s cash flow, we introduce a stochastic model of industrial economic state. The information forecasted from the state model is used as the base for adjusting the parameters of the cash flow model.

1. The characteristics of firm’s cash flow

Through our observations of cash flow, we discover that the behaviors of cash flows exhibit some stochastic characteristics, including mean-reversion and allowing positive or negative values. Figure 1-4 display these characteristics. It is understandable that a
normally managed firm tends to maintain its cash flows stable or stable with an upward trend. Since we can only acquire the quarterly data of a firm’s free cash flows, we also examine the moving-average cash flow per operating cycle (usually one year) in order to correctly demonstrate the relationship between cash flows and sales revenues (eliminating the influence management manipulation in credit policy). We discover that the trend of moving-average cash flows appears more apparent and much better matches the sales’ nature\(^9\). In sum, based on the cash flow natures we found above, a “mean-reversion stochastic process” seems appropriate to depict cash flow’s characteristics\(^0\).

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\(^9\) The relationship between sales’ natures and cash flow’s qualities will be discussed in appendix I.

\(^0\) In financial literatures, mean-reversion stochastic models are often applied to model interest rate. We therefore observe interest rate illustrated as figure A2-1 and A2-2 (refers appendix II). Comparing figures 1,2,3,4 with figures A1,A2, we discover that the mean-reversion of cash flow appear more obviously than interest rate. This is mainly because that cash flow is necessary for firm’s operation so that it is very important to maintain the level in a stable trend. On the other side, interest rate may be disturbed by many macroeconomic noises, such as the liquidity trap. When liquidity trap occurs, it will make mean-reversion disappear. So cash flow is more suitable to state by mean-reversion stochastic models.
2. Mean-reversion stochastic model

A generalized time independent mean-reversion stochastic model is stated as follows:

\[ dX = a(b - X) \cdot dt + \sigma \cdot X^\beta \cdot dz \]
\[ dz = \varepsilon \sqrt{dt} \quad \square \sim \text{N}(0,1) \quad (1) \]

where,
\( dX \) : stochastic variable X’s changes in a short time
\( a \) : stochastic variable X’s mean-reversion speed
\( b \) : the long-term average of stochastic variable X
\( \sigma \cdot X^\beta \) : the standard deviation of stochastic variable X’s changes in a short time (dt).
\( \square \) : positive constant. So, \( Var(dX) = \sigma^2 \cdot X^{2\beta} \cdot dt \)

While a generalized time-dependent mean-reversion stochastic model can be displayed as follows:

\[ dX = [\Theta(t) + a(t) \cdot (b - X)] \cdot dt + \sigma(t) \cdot X^\beta \cdot dz \quad (2) \]

In the equation (2), \( \Theta(t) \) is extra-added in drift term and it is a function of time.
In fact, equation (2) is also a kind of mean-reversion stochastic models. It can be restated as follows:

\[ dX = a(t) \cdot [b'(t) - X] \cdot dt + \sigma(t) \cdot X^\beta \cdot dz \quad (3) \]

In equation (3), \( b'(t) = ?(t) / a(t) + b \)
\( b'(t) \) stands for the long-term average of stochastic variable X varying with time;
\( a(t) \) represents for stochastic variable X’s mean-reversion speed varying with time.
Equation (3) can better describe the stochastic variable X’s fluctuating behavior because of the increase on parameters’ degrees of freedom.

We adopt the time-dependent version of mean-reverting stochastic model since a firm’s cash flows are severely influenced by changes of industrial economic states. Applying the concept of varying coefficient model\(^{21}\), the parameters in the cash flow model are time-varying to reflect the changes of future economic states. While the

\(^{21}\) It is usually applied in time-series sample data. Its characteristic is that it takes the changes of the model’s coefficients as one or one more explainable variables in another regression model. And it makes the expected value of the coefficient be decided by a series of explaining variables.
expected future economic state changes are obtained from a stochastic industrial economic state model. According to our analysis (showed in appendix III), it is appropriate to apply a time-independent mean-reverting stochastic model for modeling industrial state of economy.

3. Stochastic cash flow model

Based on above discussion, our cash flow model is a “Time-dependent stochastic cash flow model”. Since cash flow’s stochastic fluctuation nature already varies with time (namely varies with economic state), the influence of its size on cash flow’s fluctuation (the $X^0$ part in equation (1-3)) should have been reflected in the changes of economic state. We can therefore assume that $\beta$ is equal to zero in equation (3) and establish the “Time-dependent stochastic cash flow model” as equation (4):

$$dC_t = a(t) \cdot [b(t) - C_t] \cdot dt + \sigma(t) \cdot dz, \quad dz = \varepsilon \sqrt{dt}, \quad \varepsilon \sim N(0,1) \quad (4)$$

where,

d$C_t$ : cash flow’s term variation (or instantaneous changes in continuous time)
a(t) : cash flow’s mean-reversion speed,
b(t) : cash flow’s long-term average level
s(t) : standard deviation of cash flow’s term variation, namely $\sqrt{Var(dC_t)}$

“Free Cash Flow Discounted Method” has been empirically confirmed to be able to attain the true value of a firm in literature. We use free cash flow as the cash flow input in our cash flow model. The definition of free cash flow used here primarily follows that of Benninga as follows:

$$C_t = EBIT_t \cdot (1-tax rate_t) + \text{Depreciation}_t - \text{Capital Expenditures}_t - (\text{Changes in non-cash Working Capital}_t) \quad (5)$$

In equation (5), the definitions of EBIT, Capital Expenditures, and Changes in non-cash Working Capital are in the following:

EBIT = operating profits + non-operating revenue with the quality of cash flow – non-operating expense with the quality of cash flow + interest expense

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22 That is the constant parameter stochastic model shown in equation (1).
23 According to the empirical criteria of valuation methods developed by Moores International Financial Consulting Company (2004), we discover DCF method is powerful during business life cycle. Besides, real option method or APV (Adjusted Present Value) method viewed as the effective methods presently also based on DCF method. We therefore think concept of free cash flow is meaningful in enterprise’s valuation.
Capital Expenditures_t
= (Fixed Assets_{end} - Fixed Assets_{beg}) – Dep + R&D Exp. – Amortization + Acquisition

Changes in non-cash Working Capital_t
=non-cash current assets_t - non-debt current liability_t

From equation (5), we know that free cash flow is mainly influenced by EBIT, capital expenditures necessary in maintaining a stable growth, and changes of working capitals. EBIT and necessary capital expenditure are all related to sales. Because sales are primarily influenced by industrial economic state (namely business cyclical factor), there must exist a close relationship between free cash flows and economic state. Next, we will introduce the stochastic economic state model and its relationship with parameters’ adjustments of the stochastic cash flow model.

4. Stochastic economic state model and parameters’ adjustments of the stochastic cash flow model

In order to simplify our model and without loss of generalization, we assume that a(t) in equation (4) is equal to a constant\(^24\). The a(t) stand for long-run mean-reversion speed of a firm’s cash flow. While, b(t) and \(\sigma(t)\) represent for long-term average free cash flow and standard deviation of free cash flow’s term changes\(^25\) respectively. These three parameters can be estimated by Chen(1996), Miao(2003) or just simple statistics of long-term historical data.

In this study, we use a firm’s industry leading indictor\(^26\) to proxy economic state and build a stochastic industrial economic state model as equation (6) below\(^27\). With this state model, the economic states in the future periods can be estimated.

\[
d(\eta_t) = a_\eta \cdot [b_\eta - \eta_{t-1}] \cdot dt + \sigma_\eta \cdot dz \quad (6)
\]

\(^24\) Actually a(t) will be influenced by the growth trend of individual enterprise. In this paper, we assume that a(t) is a fixed constant in order to simplify model. We therefore only consider the business life cycle of individual firm when applying so that the general form of our model will be maintained.

\(^25\) In this paper, we will consider macroeconomic cycle and industrial maturity and regard them as the adjustment basis of parameters’ term-changes in stochastic cash flow model. The basic concept of this idea is that industrial maturity will influence a firm’s growth power and then a firm's growth power will also influence the future firm’s value and its variance. However, these two considerable factors will reflect on the industrial “the growth rate of coincident indictors” or “the growth rate of leading indictors”. We therefore lead the estimates of the future coincident or leading indictors’ growth rate into stochastic cash flow model and then it can be reflected on the changes of economic state (time).

\(^26\) It can be also applied in coincident indictors.

\(^27\) The characteristics of economic state (business cyclical factor) can be referred in appendix III. We discover that its fluctuation obviously has the nature of mean-reversion.
where,
\[ \eta_t : \text{the growth rate of industrial leading indicator in time } t. \]
\[ b_\eta : \text{the long-term average of industrial leading indicator’s growth rate} \]
\[ \sigma_\eta : \text{the standard deviation of the changes of industrial leading factor’s growth rate} \]

Consequently we can fine-tune the parameters of the stochastic cash flow model according to the estimates of future industrial economic states which derived from the stochastic economic state model. The parameters \( b(t) \) and \( s(t) \) in equation (4) are shown as bellow:

\[
b(t) = b \cdot (1 + \psi_t^b) \tag{7}
\]
\[
\sigma(t) = \sigma \cdot (1 + \psi_t^\sigma) \tag{8}
\]

In equations (7) and (8),
\( b \): the long-term average of cash flow calculated from historical data.
\( \sigma \): the standard deviation of cash flow’s term changes calculated from historical data.

When industrial economic state’s proxy is a leading indicator:

\[
\psi_t^b = \frac{\hat{\omega}_{t-1} - \bar{\omega}}{\bar{\sigma}} \cdot \alpha_i \tag{9}
\]
\[
\psi_t^\sigma = \frac{\hat{\omega}_{t-1} - \hat{\omega}_{t-2}}{\bar{\sigma}} \cdot |\alpha_i| \tag{10}
\]

In equation (9) and (10),
\( \hat{\omega}_i \): the estimates of industrial economic state in future periods from stochastic industrial economic state model.

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28 In our models, we assume that a firm’s cash flows reflect the state of industrial economy and then we let cash flow’s mean-reversion speed be equal to one because the cash flow model’s parameters (\( b, \bar{\sigma} \)) have been adjusted by the future industrial economic states. The details of parameters’ adjustments in stochastic cash flow model are thoroughly discussed in appendix IV (It discusses the relationship between stochastic cash flow model’s estimation of parameters and economic states and introduces the adjustment method.).

29 When economic state’s proxy is a coincident indicator:

\[
\psi_t^b = \frac{\hat{\omega}_t - \bar{\sigma}}{\bar{\sigma}} \cdot \alpha_i \quad \psi_t^\sigma = \frac{\hat{\omega}_t - \hat{\omega}_{t-1}}{\bar{\sigma}} \cdot |\alpha_i| \]
\( \overline{\omega} \): the long-term average of industrial economic state calculated from historical data

\( \alpha_1 \): the sensitivity of cash flow per unit asset relative to industrial economic state (namely the regressive coefficient of \( c_t = \alpha_0 + \alpha_1 \cdot \omega_t + \epsilon_t \), and \( c_t \) stands for the cash flow per unit asset in time \( t \)).

In the above adjustment methods, \( \alpha_1 \) reflects the sensitivity of cash flow per unit asset to the fluctuation of industrial economic state. In this study, we assume that both industrial economic state and cash flow have the same fluctuating magnitude in order to simplify model; that is to let \( \alpha_1 \) be equal to one.

On the other side, the long-term average growth rate of industrial economic state’s \( (h_b) \) and the standard deviation of changes of industrial economic state’s growth rate \( (\sigma_h) \) are both constants and are estimated by AR(1) method (Chen, 1996) \(^{30}\).

### III. The Cash Flow Based Multi-Period Credit Risk Model

A multi-period credit risk model focuses on the relationship between multi-period firm’s asset value and its liability. We first have to know the firm’s value distributions in the future periods. With our “time-dependent stochastic cash flow model”, we can simulate any probable free cash flow paths. We can then generate as many firm value paths with respect to each free cash flow path as discount rate and growth rate are reasonably estimated. As a result, we can evaluate a firm’s multi-period credit risk through the relationship between asset value distributions and debt \(^{31}\).

#### 1. Deriving a firm’s multi-period value distributions

We assume that a firm has a two-stage growth pattern. In first stage, future cash flows are simulated from our cash flow model before time \( T \) (a future time point). In second stage, cash flows grow with the constant rate \( g \) after time \( T \). Consequently we can get one future cash flow path of a firm when simulating once. With this cash flow path, we can calculate the firm’s present value at any time \( t \) following equation (11). Therefore, for each specific cash flow path, we can obtain a corresponding firm value path, too. Repeating above process for \( N \) times, we can have a firm’s \( N \) cash flow paths and also \( N \) corresponding value paths. Through a cross-sectional analysis in each period, we obtain the firm’s multi-period value distributions.

\(^{30}\) A detailed presentation of Chen’s estimation method can be seen in appendix V.

\(^{31}\) We assume that firm’s debt is fixed here to simplify our model. It can be generalized in further study by relaxing this assumption.
\[ PV_i = \frac{FCF_{i,t+1}}{(1+WACC)} + \ldots + \frac{FCF_{i,T-t+1}}{(1+WACC)^{T-t+1-1}} + \frac{FCF_{i,T-t}}{(1+WACC)^{T-t-1}} \cdot (WACC - g) \] (11)

\( PV_i \): the firm’s present value for the \( i \) th cash flow’s path at time \( t \) (the end of period \( t \)).

\( FCF_i \): the firm’s free cash flow for the \( i \) th cash flow’s path at time \( t \).

\( T \): the beginning time of constant growth.

\( WACC \): firm’s weighted average costs of capital

\( g \): the estimate of constant growth.

In equation (11), \( g \) and \( WACC \) significantly influence a firm’s present value distribution.

Regarding the WACC, we calculate it by using equation (12).

\[ WACC = \frac{D}{A} \cdot r_g \cdot (1-t) + \frac{E}{A} \cdot r_e \]
\[ r_e = r_f + \beta \cdot (r_m - r_f) \] (12)

In equation (12), the parameters \( r_e \) and \( r_d \) play important roles. In this study, we assume that they are fixed constants to simplify model32.

Regarding the firm cash flow’s constant growth rate after time, we employ two common used methods in this study. First, we use the \( g \) usually appearing in corporate finance textbook, calculating as \( g = ROIC \cdot RI \); Where ROIC stands for the long-term average return rate of investment capital, and RI represents for the long-term average of re-investment rate. Second, we calculate the average growth rate of industrial leading indictors as the estimate of the \( g \). To choose between the two estimates of \( g \), we introduce time series analysis of the sales per unit asset. We pick the one that better fit the trend of the firm’s sales per unit asset through its PACF (Partial Autocorrelation Function).

Having the WACC and growth rate, we can implement discount method and obtain multi-period assets value distributions.

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32 In this study, to determine \( r_e \), the estimation of Taiwan’s market risk premium( \( r_m - r_f \) ) are based on the American’s market risk premium. We calculate the returns volatility relationship between these two markets and use the concept of CML (Capital Market Line) to derive Taiwan’s reasonable market risk premium. Through CAPM theory, we can calculate \( r_e \) when other variables (namely risk-free rate and beta) can be also reasonably estimated.
2. Deriving a firm’s multi-period value distributions

In this study, we use long-term liabilities as the proxy for critical point of default risk. This is because we are building a multi-period credit risk model and long-term liabilities are the main source of credit risk. Long-term liabilities are usually planned and have clear schedules (such as the issue of corporate bonds, financing debts) so that it is easy to estimate in a long run perspective. Current liabilities, on the other side are difficult to estimate because it is influenced by a firm’s credit policy and is usually more volatile. It is appropriate that we take long-term liabilities as critical point for default determination because we assume that liabilities are fixed in this study. The default determination can be illustrated as figure 5.

![Figure 5. Default risk determination method](image)

In figure 5, default will occur when Firm’s present value is less than loan balance and probability of default can be showed as below:

\[
\text{Probability of default} = \int_{-\infty}^{\infty} f(PV_i) \cdot d(PV_i) \quad (13)
\]

The $L$ in above equation indicates the liabilities (or loan) balance of the firm. It is used here as critical point for default determination.

For creditors, the expected default loss is stated as equation (14).

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33 There are different opinions on the decision of liability when measuring credit risk. For example, KMV mentions “0.5*current liability + long-term liability” to be the critical.
Expected Loss = $\bar{L} \cdot \int_{-\infty}^{\tau} f(PV_i) \cdot d(PV_i) - \int_{0}^{\tau} PV_i \cdot f(PV_i) \cdot d(PV_i)$ \hspace{1cm} (14)

For creditors, the loss given default (later denoted as LGD) can be showed as equation (15).

\[
LGD = \frac{1}{PD} \cdot \left[ \bar{L} \cdot \int_{-\infty}^{\tau} f(PV_i) \cdot d(PV_i) - \int_{0}^{\tau} PV_i \cdot f(PV_i) \cdot d(PV_i) \right] 
= \bar{L} - \frac{1}{PD} \cdot \int_{0}^{\tau} PV_i \cdot f(PV_i) \cdot d(PV_i)
\] (15)

Therefore, creditor’s recovery rate (RR) when default occurs can be written as equation (16):

\[
RR = \frac{1}{L} \left[ \bar{L} - \bar{L} + \frac{1}{PD} \cdot \int_{0}^{\tau} PV_i \cdot f(PV_i) \cdot d(PV_i) \right] = \frac{1}{L \cdot PD} \cdot \int_{0}^{\tau} PV_i \cdot f(PV_i) \cdot d(PV_i)
\] (16)

In current study, we define a new variable, expected recovery rate (later denoted as ERR), which is equal to one minus the ratio of expected loss to loan balances. It means how much that creditors can expect to recover their loans. ERR can be written as equation (17):

\[
ERR = 1 - \frac{\bar{L} \cdot \int_{-\infty}^{\tau} f(PV_i) \cdot d(PV_i) - \int_{0}^{\tau} PV_i \cdot f(PV_i) \cdot d(PV_i)}{\bar{L}} 
= 1 - PD + \frac{\int_{0}^{\tau} PV_i \cdot f(PV_i) \cdot d(PV_i)}{\bar{L}}
= 1 - PD + \frac{RR \cdot \bar{L} \cdot PD}{\bar{L}} = 1 \cdot (1 - PD) + RR \cdot PD
\] (17)

Therefore, we can make the two main credit risk factors, PD and RR, be endogenous in our model based on the above discussion. Besides, we also discover that PD and RR are inversely related and PD and ERR are negatively related.

Regarding credit risk of specific firm obligations (e.g. corporate bonds), the seniority of the obligations becomes crucial. If there are other debts senior to the debt we are assessing, we have to deduct those senior debt balances from the firm’s asset value before doing above credit analysis. When there are other liabilities with the same seniority as the debt we are considering, all those debts have to be added into $\bar{L}$. The default

\[34\] The inverse relationship between PD and RR will be discussed in appendix VI.
A firm’s debt composition might change in the future. Whether these changes affect the credit of the specific debt we are interested in depends on whether they affect the PD of the specific debt. For example, the credit of specific debt will improve if the firm borrows subordinate debt because the firm’s asset value increases. Another example is that the specific debt’s credit becomes worsen if there is an unexpected tax charges by government because tax liabilities increases but asset value doesn’t.

In summary, the design process of our “Cash flow based multi-period credit risk model and its pricing method” can be illustrated as figure 6.

Figure 6. Flow Chart of the CF-Based Multi-period Credit Risk Model and Application
IV. Empirical Analysis

In order to examine the effectiveness of our “Cash Flow Based Multi-period Credit Risk Model”, we select 24 companies from Taiwan’s stock market to analyze their credit risks. Moreover, we also use UMC’s unsecured corporate bonds as an example to show the application of our model in pricing corporate specific debts.

1. Data

Because this study develops three models, “Stochastic industrial economic state model”, “Cash flow based multi-period credit risk model”, and “Bond Pricing model with the concept of J-T model”, we have to get all related data to do the empirical analysis. First, we establish the applicable proxies for industrial economic states according to the criteria of NBER (in America) or Council for Economic Planning and Development (in Taiwan) and they are illustrated in table 1.

Second, we select 24 rated firms that are rated by either Standard and Poor’s or Taiwan Ratings Corporation (later denoted as TRC) as our samples. Names of these sample companies are shown in table 5. These sample companies’ industrial categories are illustrated in table 2.

Third, we get individual company’s financial data to calculate free cash flow and loan balance from TEJ database so that we can complete the work of evaluating individual company’s multi-period credit risk. Forth, we get data of YTM for 2-year, 5-year, and 10-year on-the-run issues from Gretai Securities Market (an organization of Taiwan’s OTC Market) for simulating an affine term structure of Taiwan’s YTM. Fifth, we proceed our “Bond Pricing Model with the concept of J-T model” by using the UMC’s unsecured corporate bond data and an affine term-structure of Taiwan’s YTM. To sum up, all the data sources are completely illustrated in table 3.

The estimation period of business cyclical factor (industrial economic state) and free cash flow per unit asset is from 1995 to 2003Q2.

35 In order to simulate the future PV distributions, we have to estimate the individual company’s WACC and its growth rate. On the part of WACC, we estimate Taiwan’s risk premium based on the CML theory. According to the return indexes (RI) both of American and Taiwan’s markets from datastream, we can separately calculate the RI’s annualized volatility of America (0.1815) and Taiwan (0.3393). Moreover, we can get American annualized risk premium data (7.5%) from “Ibboson Associated, annual”. After transforming by exchange rate, we can finally estimate Taiwan’s annualized risk premium (14.03%). On the other part of growth rate, we utilize the method of financial theory (g=ROIC*RI) or industrial growth rate.

36 BB ratio is the exception. Because of its difficulty to acquiring the more past data, we can estimate the parameters from 1998Q2 to 2003Q2. But the data of telecom industry can be acquired in a shorter period, that is from 2000 to 2003Q2.
Table 1: The applicable proxies for industrial economic state

<table>
<thead>
<tr>
<th>Leading indicators</th>
<th>sources</th>
<th>Coincident indicators</th>
<th>sources</th>
</tr>
</thead>
<tbody>
<tr>
<td>The change rate of each industrial stock index</td>
<td>TEJ,</td>
<td>1. each industrial sales revenues</td>
<td>TEJ,</td>
</tr>
<tr>
<td></td>
<td>Datastream</td>
<td></td>
<td>Datastream</td>
</tr>
<tr>
<td>The change rate of each industrial added-orders</td>
<td>TEJ,</td>
<td>2. the change rate of each industrial production</td>
<td>TEJ,</td>
</tr>
<tr>
<td></td>
<td>Datastream</td>
<td></td>
<td>Datastream</td>
</tr>
<tr>
<td>For specified industry (e.g. Semiconductor, DRAM) with compiled index or goods' price</td>
<td>SEMI,</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bloomberg,</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Datastream</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* The decisions of leading indictors or coincident indicators primarily depend on the business cyclical indicators selected by Council for Economic Planning and Development (in Taiwan)

** The decisions of American macroeconomic indicators are similar with Taiwan.

Table 2: The industrial categories’ distribution of empirical sample

<table>
<thead>
<tr>
<th>Industry</th>
<th>Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plastics</td>
<td>3</td>
</tr>
<tr>
<td>Iron &amp; Steel</td>
<td>1</td>
</tr>
<tr>
<td>Auto &amp; Truck Manufacturers</td>
<td>1</td>
</tr>
<tr>
<td>Electronics</td>
<td>14</td>
</tr>
<tr>
<td>Telecom</td>
<td>3</td>
</tr>
<tr>
<td>Construction</td>
<td>1</td>
</tr>
<tr>
<td>Airline</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 3: All related empirical data sources

<table>
<thead>
<tr>
<th>Items</th>
<th>Corporate financial data and ratings</th>
<th>Industrial economic state (business cyclical factors)</th>
<th>Bonds &amp; YTM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sources</td>
<td>TEJ, TRC</td>
<td>TEJ, SEMI, Datastream</td>
<td>Gretai Securities Market</td>
</tr>
</tbody>
</table>

2. Parameters’ estimation of stochastic industrial economic state model

To adjust the parameters of our cash flow model, we have to know the industrial economic state model first. Therefore we have to estimate the parameters of the industrial economic state model. Here we select some business cyclical factors (industrial economic state) and let the change rate of each industrial stock index be a proxy for
industrial economic state except semiconductor industry. We use B-B ratio as the proxy for semiconductor industry because it can more correctly reflect the changes of industrial economic state (business cycle). We employ AR(1) method (Chen, 1996) to estimate its parameters and the results can be illustrated in table 4.

Table 4. Parameters’ estimation of stochastic industrial economic state model

<table>
<thead>
<tr>
<th>Parameters’ estimation of stochastic industrial economic state model</th>
<th>Industry</th>
<th>Plastics</th>
<th>Iron &amp; Steel</th>
<th>Auto &amp; Truck Manufacturers</th>
<th>Electronics</th>
<th>Semiconductor</th>
<th>Construction</th>
<th>Airline</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulation target</td>
<td>Change rate of Plastics stock index</td>
<td>Change rate of Iron &amp; Steel stock index</td>
<td>Change rate of Auto &amp; Truck Manufacturers stock index</td>
<td>Change rate of Electronics stock index</td>
<td>Change rate of Book-to-Bill ratio</td>
<td>Change rate of construction stock index</td>
<td>Change rate of Airline stock index</td>
<td></td>
</tr>
<tr>
<td>Simulation target</td>
<td>a</td>
<td>1.7779</td>
<td>2.1349</td>
<td>1.4396</td>
<td>1.3823</td>
<td>1.3053</td>
<td>2.1085</td>
<td>1.5929</td>
</tr>
<tr>
<td>Simulation target</td>
<td>b</td>
<td>0.0166</td>
<td>-0.0101</td>
<td>0.0587</td>
<td>-0.0135</td>
<td>0.0077</td>
<td>-0.0076</td>
<td>-0.0085</td>
</tr>
<tr>
<td>Simulation target</td>
<td>s</td>
<td>0.1141</td>
<td>0.1589</td>
<td>0.2052</td>
<td>0.0712</td>
<td>0.1432</td>
<td>0.0420</td>
<td>0.0469</td>
</tr>
</tbody>
</table>

3. Empirical results of firms’ credit ratings

The empirical credit analyses results are illustrated in table 5. The sixth column of table 5 stands for probability of default of each sample firm calculated by our credit risk model (denoted as “model’s PD”) during the future ten years. The fifth column of table 5 represents for each firm’s theoretical rating (denoted as “model’s rating”). The model’s ratings are assigned to each firm by comparing model’s PD to the ten-year cumulative default rates curve in American market. Hence, the model’s ratings are equivalent to American (or global) rating. Since Taiwan market is essentially not the same as American market in terms of having different risk factors, such as political risk, country risk and so on, our model’s rating should be lower than the firm’s local rating given by TRC (shown in the fourth column of table 5).

According to empirical results illustrated in table 5, we discover that 19 firms are downgraded for 1/3~1 rating grade from TRC rating, 3 firms are rated the same as TRC,

---

37 Due to the limitation of acquiring data, we can only select “the change rate of each industrial stock index” easier to be acquired to stands for industrial economic state.

38 The cumulative default rate curve is provided by Standard and Poor’s (1981~2002).

39 In practice, a rule of thumb for the rating difference is that Taiwan local ratings are about one rating grade lower than those of Global rating. For example, in practice a twA- rating is equivalent to a global rating BBB-. From model’s perspective, Taiwan’s cumulative default rates curve should add a country rating spread to be equivalent to that of global (or American) market.
and 2 firms are upgraded 1/3-1 rating grade from TRC rating. From these results, 19 out of 24 firms (about 80%) are rated reasonably. Our model’s effectiveness seems preliminarily supported by the empirical evidences. Among the 24 firms, two firms are upgraded by our model, the Foxconn and TSMC. From our observation, these two firms have very stable and abundant free cash flow so that their default probabilities are much lower than other firms. TRC may consider other factors rather than just their operating performance, such as lawsuits, significant strategic investment activities with high uncertainty. It is our current model’s constraint that we cannot consider all the uncertainty in our model. It could be improved by adding extra stochastic terms into our model, such as “jump diffusion model” to take care of more uncertainties.

4. Example of corporate bond pricing

Regarding our model’s application in corporate bond pricing, we take UMC’s unsecured corporate bond issued in 2003 as our example\textsuperscript{40}. We adopt Chen’s AR(1) estimation method to estimate parameters in the stochastic industrial economic state model and the results are illustrated in table 6\textsuperscript{41}. The simulated multi-period distributions of both UMC’s free cash flows and present values are illustrated in figure 7 and 8. After deciding UMC’s future ERR (expected recovery rate, the expect value of recovery per unit debt), we can proceed pricing on UMC’s unsecured corporate bond with the concept of J-T model\textsuperscript{42}. The pricing results\textsuperscript{43} of UMC’s unsecured corporate bond are illustrated in table 7.

According to table 7, we find that there existed premium for UMC’s corporate bond. This is mainly because UMC's rating was high and the coupon rate was higher than market rate when issued. This example shows our model can help not only calculate theoretical price of corporate bond but also provide a trading basis for secondary corporate bond market.

\textsuperscript{40} This UMC’s unsecured bond has the characteristics as follows: five-year bond and annualized coupon rate is 2.0%.
\textsuperscript{41} Actually we employ three different estimation methods and found that Chen’s method provides the best estimates. The results of the other two methods are shown below:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>a</th>
<th>b</th>
<th>σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Historical long-term average</td>
<td>0.0009</td>
<td>0.0262</td>
<td>0.2932</td>
</tr>
<tr>
<td>Miao(2003)</td>
<td>0.7290</td>
<td>0.3085</td>
<td></td>
</tr>
</tbody>
</table>

\textsuperscript{42} J-T model is based on zero-coupon government (default-free) bond. We can let N-year coupon bond(pay once per year) be divided into N’s zero coupon bond without default risk and can further calculate the prices of the N’s defaultable zero coupon bonds through ERR. And then we can get the probable price of the defaultable coupon bond by summarizing the prices of N’s defaultable zero coupon bond.

\textsuperscript{43} Because there is no complete YTM term structure in Taiwan, we utilize the on-the-run issued government bonds to construct a simulated yield curve.
Table 5. Empirical results of cash flow based multi-period credit risk model

<table>
<thead>
<tr>
<th>Item</th>
<th>Company</th>
<th>Time</th>
<th>Actual rating</th>
<th>Model’s rating</th>
<th>Model’s PD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>CHT***</td>
<td>2003/12/15</td>
<td>twAAA</td>
<td>AAA</td>
<td>0.26%</td>
</tr>
<tr>
<td>2</td>
<td>Fareastone*</td>
<td>2003/1/27</td>
<td>twA+</td>
<td>BBB+</td>
<td>6.88%</td>
</tr>
<tr>
<td>3</td>
<td>TCC*</td>
<td>2003/3/18</td>
<td>twA+</td>
<td>A-</td>
<td>3.67%</td>
</tr>
<tr>
<td>4</td>
<td>CSC*</td>
<td>2003/6/30</td>
<td>twAA+</td>
<td>AA-</td>
<td>1.43%</td>
</tr>
<tr>
<td>5</td>
<td>UMC*</td>
<td>2002/3/15</td>
<td>twAA-</td>
<td>A</td>
<td>2.24%</td>
</tr>
<tr>
<td>6</td>
<td>TSIC**</td>
<td>2003/3/13</td>
<td>twAA+</td>
<td>AAA</td>
<td>0.04%</td>
</tr>
<tr>
<td>7</td>
<td>FPC*</td>
<td>2003/12/22</td>
<td>twAA-</td>
<td>A+</td>
<td>1.75%</td>
</tr>
<tr>
<td>8</td>
<td>NPC*</td>
<td>2003/12/4</td>
<td>twAA-</td>
<td>A</td>
<td>1.93%</td>
</tr>
<tr>
<td>9</td>
<td>Foxconn**</td>
<td>2004/3/22</td>
<td>twAA-</td>
<td>AAA-</td>
<td>0.72%</td>
</tr>
<tr>
<td>10</td>
<td>Compal*</td>
<td>2003/6/11</td>
<td>twA+</td>
<td>BBB</td>
<td>8.26%</td>
</tr>
<tr>
<td>11</td>
<td>Quanta*</td>
<td>2003/6/13</td>
<td>twA+</td>
<td>A</td>
<td>2.52%</td>
</tr>
<tr>
<td>12</td>
<td>SPIL*</td>
<td>2004/2/23</td>
<td>twBBB+</td>
<td>BBB</td>
<td>7.55%</td>
</tr>
<tr>
<td>13</td>
<td>NTI*</td>
<td>2004/2/27</td>
<td>twBBB+</td>
<td>BBB-</td>
<td>16.37%</td>
</tr>
<tr>
<td>14</td>
<td>Cmcent*</td>
<td>2003/12/18</td>
<td>twBBB</td>
<td>BBB-</td>
<td>12.34%</td>
</tr>
<tr>
<td>15</td>
<td>RITEK***</td>
<td>2003/9/16</td>
<td>twBB+</td>
<td>BB+</td>
<td>18.14%</td>
</tr>
<tr>
<td>16</td>
<td>Yageo*</td>
<td>2003/12/16</td>
<td>twBB+</td>
<td>BB</td>
<td>26.79%</td>
</tr>
<tr>
<td>17</td>
<td>QDI*</td>
<td>2004/2/17</td>
<td>twBB+</td>
<td>BB</td>
<td>23.32%</td>
</tr>
<tr>
<td>18</td>
<td>MXIC***</td>
<td>2003/5/28</td>
<td>twBB</td>
<td>BB</td>
<td>26.39%</td>
</tr>
<tr>
<td>19</td>
<td>BENQ*</td>
<td>2003/5/29</td>
<td>twA-</td>
<td>BBB</td>
<td>9.66%</td>
</tr>
<tr>
<td>20</td>
<td>WWEI*</td>
<td>2003/5/29</td>
<td>twBBB</td>
<td>BB+</td>
<td>20.61%</td>
</tr>
<tr>
<td>21</td>
<td>Cathay-red*</td>
<td>2003/11/3</td>
<td>twBBB</td>
<td>BBB-</td>
<td>15.54%</td>
</tr>
<tr>
<td>22</td>
<td>China-airline*</td>
<td>2003/11/13</td>
<td>twBBB</td>
<td>BB+</td>
<td>18.20%</td>
</tr>
<tr>
<td>23</td>
<td>FCFC*</td>
<td>2003/12/4</td>
<td>twAA-</td>
<td>A-</td>
<td>3.93%</td>
</tr>
<tr>
<td>24</td>
<td>Yulon-motor*</td>
<td>2004/2/25</td>
<td>twA</td>
<td>BBB</td>
<td>9.67%</td>
</tr>
</tbody>
</table>

*: with downgrade during 1~3 ranges; **: with upgrade during 1~3 ranges; ***: the same rating

Model’s PD: Under the assumption of homogenous markets, we can let model’s PD correspond to American ten-year cumulative default rates curve.

Table 6. The parameters’ estimation of stochastic industrial economic state model under different methods

<table>
<thead>
<tr>
<th>Parameter</th>
<th>a</th>
<th>b</th>
<th>(\sigma)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate</td>
<td>1.3053</td>
<td>0.0077</td>
<td>0.1432</td>
</tr>
</tbody>
</table>
Table 7. The Pricing of UMC’s unsecured corporate bond

<table>
<thead>
<tr>
<th>Year</th>
<th>Zero Yield</th>
<th>(d(0,t))</th>
<th>ERR</th>
<th>(V(0,t))</th>
<th>(C)</th>
<th>(P(0,t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.91%</td>
<td>0.9910</td>
<td>1.0000</td>
<td>0.9910</td>
<td>2.0</td>
<td>1.9819</td>
</tr>
<tr>
<td>2</td>
<td>0.95%</td>
<td>0.9813</td>
<td>1.0000</td>
<td>0.9813</td>
<td>2.0</td>
<td>1.9627</td>
</tr>
<tr>
<td>3</td>
<td>1.04%</td>
<td>0.9694</td>
<td>1.0000</td>
<td>0.9694</td>
<td>2.0</td>
<td>1.9388</td>
</tr>
<tr>
<td>4</td>
<td>1.14%</td>
<td>0.9558</td>
<td>1.0000</td>
<td>0.9558</td>
<td>2.0</td>
<td>1.9115</td>
</tr>
<tr>
<td>5</td>
<td>1.23%</td>
<td>0.9406</td>
<td>0.9998</td>
<td>0.9404</td>
<td>102.0</td>
<td>95.9262</td>
</tr>
</tbody>
</table>

\(\sum P(0,t) = 103.7211\)

d\((0,t)\): discounted factor of zero-coupon bond without default risk
\(V(0,t)\): the value of zero-coupon bond with default risk / per face value
\(C\): coupon per one hundred
\(P(0,t)\): the value of zero-coupon bond with default risk / per face value in hundred
\(\sum P(0,t)\): the value of coupon bond with default risk / per face value in hundred

Figure 7. UMC’s Multi-period Free Cash Flow distributions
V. Conclusions

The development of credit risk models grows rapidly recently but mainly on the reduced-form models. However, reduced-form credit models depend on specific exogenous information such as credit rating and recovery rate. They cannot generate credit risk variables such as PD, RR and LGD internally from a firm’s operation value. Besides, the developments of multi-period credit risk models are few in literatures. This study tries to fill up this literature gap.

“Cash Flow Based Multi-period Credit Risk Model” constructed in this study provides a systematic measuring process of credit risk. It starts from determining a firm’s future value distributions by our “Time-dependent stochastic cash flow model” and then combining value distributions with liability information to perform a multi-period credit risk assessment. It can also combine those endogenously generated PD and RR with J-T Model to price multi-period debts. One of major merits of our models is that they can directly price the credit risk of debts without knowing the firms’ credit rating and they straightly consider the firm’s future operating values to do multi-period credit risk analyses instead of a backward solution from firm’s credit rating. For outside investors and people inside a firm, our study provides a multi-period credit risk model that needs only publicly available information of both corporate finance and the industrial economic state (i.e. the industrial cyclical information). We believe this “Cash Flow Based Multi-period Credit Risk Model” has provided a new way for analyzing corporate debts.

Figure 8. UMC’s Multi-period Present Value distributions
Reference


27. KOU, STEVEN G., 2002, ”A Jump Diffusion Model For Option Pricing”, *Working paper*


Appendix I. The explorations of the relationship between sales’ and cash flow’s natures

In this paper, we use the quarterly financial data when estimating the parameters of stochastic cash flow model. So the calculated quarterly free cash flow will fluctuate greatly caused by companies’ credit policy so that it can’t completely fit in with the natures of real sales. In order to smooth the cash flow’s huge fluctuation, we will let historical cash flows be moving average by its operating cycle (assume one year). This rolling method will still maintain the cycle factor of industrial economic state and exclude the influence of credit policy on free cash flow. In this way, the moving average free cash flow will exactly fit in with the changes of industrial economic states so that we can reasonably utilize the “stochastic industrial economic state model” to adjust the future estimates of the “time-dependent stochastic cash flow model”.

In the following, we will introduce the natures (namely its trend and cycle) of sales per assets, original free cash flow per assets and moving average free cash flow per assets by using time-series analysis and then we take TSMC as example. Figure A1-1, A1-2 individually stands for the cycle and the trend of TSMC’s quarterly sales per assets through ACF (Autocorrelation Function) and PACF (Partial Autocorrelation Function). Figure A1-3, A1-4 separately stands for the cycle and the trend of TSMC’s quarterly original free cash flow per assets. Figure A1-5, A1-6 individually represents for the cycle and the trend of TSMC’s quarterly moving average free cash flow per assets. Through the observations of figures A1-1, A1-2, A1-3, A1-4, A1-5 and A1-6, we discover that the natures of original free cash flow per assets don’t fit in with the natures of sales per assets (sales’ nature is ARMA(1,2); original free cash flow’s nature is ARMA(0,0)); on the opposite, the moving average free cash flow per assets matches up with sales’ natures (moving average free cash flow’s nature is ARMA(1,2)).

Moreover, we will also discuss what causes original free cash flow’s nature is ARMA(0, 0). In original free cash flow per assets, its cycle can be divided into two influencing factors. One is industrial economic state, and the other is credit policy during the operating cycle. However, these two factors are negatively related⁴⁴ so that the characteristics of original free cash flow per assets will disappear.

---

⁴⁴ When industrial economic state is boom, companies’ credit policy will be relaxed.
Figure A1-1. ACF of TSMC’s quarterly sales (is used to judge “cycle”)

Figure A1-2. PACF of TSMC’s quarterly sales (is used to judge “trend”)

###Autocorrelation Function for 2330

<table>
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###Partial Autocorrelation Function for 2330

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Figure A1-3. ACF of TSMC’s original quarterly FCF (is used to judge “cycle”)

Figure A1-4. PACF of TSMC’s original quarterly FCF (is used to judge “trend”)
Figure A1-5. ACF of TSMC’s moving-average quarterly FCF  
(is used to judge “cycle”)

Figure A1-6. PACF of TSMC’s moving-average quarterly FCF  
(is used to judge “trend”)

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Appendix II. The historical trend of interest rates

![CP2 historical trend in Taiwan](image1)

Figure A2-1. CP2 historical trend in Taiwan\(^45\)

![6-month LIBOR historical trend](image2)

Figure A2-2. 6-month LIBOR historical trend\(^46\)

\(^45\) Sources: TEJ database
\(^46\) Sources: http://www.economagic.com/libor.htm#US
Appendix III. The stochastic characteristics of industrial economic state

In this paper, we use the change rate of each industrial stock index to be the proxies for industrial economic state factors and then observe the trends (the sample period is from 1995 to 2003). These industrial categories include plastics, iron & steels, auto manufactures, and electronics. Moreover, we also consider the trend of book-to-bill ratio (B/B) for semiconductors. The historical trend of each industrial economic state factor is illustrated as the following figures and we discover that there exists the phenomenon of mean-reversion in all industrial categories. We therefore think industrial economic state applicable to mean-reversion stochastic model.

![Figure A3-1. Plastics stock index’s %M](image1)

![Figure A3-2. Iron & Steel stock index’s %M](image2)

![Figure A3-3. Auto Manu. stock index’s %M](image3)

![Figure A3-4. Electronics stock index’s %M](image4)
Trend Analysis for $BB$
Linear Trend Model
$Y_t = 1.03506 - 1.35E-03*t$
MAPE: 24.7847
MAD: 0.2074
MSD: 0.0623

Trend Analysis for $\%BB$
Linear Trend Model
$Y_t = 2.48E-02 - 3.25E-04*t$
MAPE: 111.541
MAD: 0.078
MSD: 0.011

Figure A3-5. $BB$ %M
Figure A3-6. $BB$%M
Appendix IV. The method to estimate parameters of time-dependent stochastic cash flow model

In this paper, our stochastic cash flow model can be showed as equation (A4-1):

\[ dC_t = a(t) \cdot [b(t) - C_t] \cdot dt + \sigma(t) \cdot dz \]

\[ dz = \varepsilon \sqrt{dt} \quad \varepsilon \sim N(0,1) \quad (A4-1) \]

In equation (A4-1):
- \( dC_t \): the term changes of free cash flow
- \( a(t) \): the mean-reversion speed of free cash flow
- \( b(t) \): the long-term average of free cash flow
- \( s(t) \): the standard deviation of the term changes of free cash flow, namely \( \sqrt{Var(dC_t)} \)

In the estimation of stochastic cash flow model’s parameters (\( a, b, \sigma \)), we can calculate the fixed constant by using non-parameter method (Miao, 2003), AR(1) method (Chen, 1996), or generalized method of historical data.

Now we want to let \( b \) and \( \sigma \) be time-varying so that we utilize stochastic industrial economic state model to make adjustments. In the following, we let \( c \) stand for cash flow per unit asset and \( w \) stand for the industrial economic state factor. Further we explore the relationship between cash flow (\( c \)) and industrial economic state factor (\( w \)) by constructing the regression showed as equation (A4-2):

\[ c_t = \alpha_0 + \alpha_1 \cdot (w_t) \quad (A4-2) \]

So we can make time-varying adjustments on the long-term average of cash flow (\( b(t) \) ) based on the future cash flow’s growth rate relative to \( b \). According to equation (A4-2), we can further transfer the future cash flow’s growth rate to the future industrial economic state factor’s growth rate:

\[ b(t) = b \cdot (1 + \frac{c(t) - \bar{c}_{long-term}}{\bar{c}_{long-term}}) \]

\[ = b \cdot (1 + \frac{\omega_t - \bar{\omega}}{\bar{\omega}}) \cdot \alpha_i \quad (A4-3) \]

\[ = b \cdot (1 + \alpha_i \cdot (\frac{\omega_t - \bar{\omega}}{\bar{\omega}} - 1)) \]

In equation (A4-3), \( \frac{\omega_t - \bar{\omega}}{\bar{\omega}} \) stands for the future industrial economic state factor’s growth rate relative to the long-term average of industrial economic state factor (\( \bar{\omega} \)).
Moreover, we have to consider the regressive coefficient $\alpha_1$ when making adjustments on the long-term average of cash flow ($b(t)$) according to equation (A4-3).

In addition, we will discuss the method to make the variances of cash flow ($\sigma$) be time-varying. First, we will difference on the both sides of equation (A4-2) and then take variances on the difference results. Second, we will explore the relationship between “the effect on the changes of $\Delta(c_i)$ caused by the changes of $\Delta(\omega_i)$” and “the effect on the $\sigma_{\Delta(c_i)}$ caused by the changes of $\sigma_{\Delta(\omega_i)}$”. At last, we can infer the adjustment methods for the variances of cash flow ($\sigma$). In the following, we will display our inference:

Difference on both sides of equation (A4-2) as follows:

$$\Delta(c_i) = \alpha_1 \cdot \Delta(\omega_i) \quad (A4-4)$$

Take variances on both sides of equation (A4-4) as follows:

$$\text{Var}(\Delta(c_i)) = \alpha_1^2 \cdot \text{Var}(\Delta(\omega_i)) \Rightarrow \sigma_{\Delta(c_i)} = |\alpha_1| \cdot \sigma_{\Delta(\omega_i)} \quad (A4-5)$$

So we can summarize as equation (A4-6) when $\alpha_1$ is a positive constant according to equation (A4-4) and (A4-5):

$$\alpha_1 = \frac{\Delta(c_i)}{\Delta(\omega_i)} = \frac{\sigma_{\Delta(c_i)}}{\sigma_{\Delta(\omega_i)}} \quad (A4-6)$$

And we can also get equation (A4-7) when $\alpha_1$ is a negative constant

$$-\alpha_1 = -\frac{\Delta(c_i)}{\Delta(\omega_i)} = \frac{\sigma_{\Delta(c_i)}}{\sigma_{\Delta(\omega_i)}} \quad (A4-7)$$

We can therefore conclude that the size of “effect on the changes of $\Delta(c_i)$ caused by the changes of $\Delta(\omega_i)$” (called $A$ event) will be the same with the size of “effect on the $\sigma_{\Delta(c_i)}$ caused by the changes of $\sigma_{\Delta(\omega_i)}$” (called $B$ event) when industrial economic state changes in the future. But it is necessary to notice that the relationship of these two events will vary with $\alpha_1$; that is to say, the direction of $A$ event will be opposite to the direction of $B$ event when $\alpha_1$ is negative.
In equations (A4-6) and (A4-7), we can know that \( \sigma_{\Delta(c_r)} \) is a function of \( \sigma_{\Delta(\omega_r)} \) and \( \Delta(c_r) \) is a function of \( \Delta(\omega_r) \). And both two functions are estimated the same base, namely \( \alpha_1 \) which is the regressive coefficient in \( \Delta(c_r) = \alpha_1 \cdot \Delta(\omega_r) \). Therefore according to the concept of varying coefficient model, the effects on \( A \) event and \( B \) event will be the same by \( \alpha_1 \) when the industrial economic state changes in the future \(( \Delta(\omega_r), \sigma_{\Delta(\omega_r)} )\). As a result, we can make adjustments on the variances of cash flow(\( \sigma \)) by using \( A \) event instead of \( B \) event. In addition, we can assume that \( \alpha_1 \) is equal to one. It is because we think that the changes of industrial economic state will completely reflect on a firm’s cash flow changes\(^{47,48}\). In the following, we will infer the \( A \) event’s effect firstly, then apply the result in \( B \) event and at last we can conclude the adjustment methods of the variances of cash flow(\( \sigma \)):

**Inferences:**

When the industrial economic state factor is \( \omega_r \) in the future time \( t \), we can get the adjustment effect of reflecting on the long-term average cash flow \( b \) according to equation (A4-3):

\[
\frac{\Delta b(t)}{b} = \alpha_1 \cdot \left( \frac{\omega_r}{\bar{\sigma}} - 1 \right)
\]  

(A4-8)

When the industrial economic state factor is \( \omega_r \) in the future time \( t+1 \), we can get the adjustment effect of reflecting on the long-term average cash flow \( b \) according to equation (A4-3):

\[
\frac{\Delta b(t+1)}{b} = \alpha_1 \cdot \left( \frac{\omega_{r+1}}{\bar{\sigma}} - 1 \right)
\]  

(A4-9)

So we make equation (A4-8) minus equation (A4-9) and then get the influencing amount of \( A \) event:

---

\(^{47}\) This assumption is primarily from strong economic intuition. In the future, we can try to find some variables with great influencing on a firm’s cash flow and then composite the new factor which can explain the changes of a firm’s cash flow.

\(^{48}\) On the other side, our methods fit in with the concept of varying coefficient model”.

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\[ b \cdot \alpha_1 \cdot \left( \frac{\omega_{i+1} - \omega_i}{\alpha} \right) \quad (A4-10) \]

Therefore the influencing size of \textit{A event} will be showed as the equation (A4-11) when the base of cash flow is b:

\[ \alpha_1 \cdot \left( \frac{\omega_{i+1} - \omega_i}{\alpha} \right) \quad (A4-11) \]

According to equation (A4-11), we can know that cash flow will change by the rate of \( \alpha_1 \cdot \left( \frac{\omega_{i+1} - \omega_i}{\alpha} \right) \) with the varying of industrial economic state. Moreover, we can know that \textit{A event} has the effect with \textit{B event} from equation (A4-6) and (A4-7). But it is necessary to notice that \( \alpha_i \) have to be taken as its absolute value.

In this paper, we assume that the changes of industrial economic state will fully be reflected on the changes of firms’ firm on normal operation excluding discretionary expenditures. So we can assume that \( \alpha_i \) is equal to one. However, cash flow may probably be explained by other variables so we will try to create a new factor with high exploratory power for cash flow by using factor analysis methods. Further we can get the better result.
Appendix V. Chen’s (1996) AR(1) method: apply in stochastic industrial economic state model

In this paper, we construct the stochastic industrial economic state (∑,) model showed as (A5-1):

\[d(\eta_t) = a_\eta \cdot [b_\eta - \eta_{t-1}] \cdot dt + \sigma_\eta \cdot dz\]  \hspace{1cm} (A5-1)

In equation (A5-1), both the parameters, \(b_\eta\) and \(\sigma_\eta\), are estimated by using the AR(1) method (Chen, 1996). \(b_\eta\) stands for the long-term average change rate of industrial economic state and \(\sigma_\eta\) stands for the standard deviation of the changes of industrial economic state’s fluctuating rate.

Chen’s estimate method is under the O-U process, and the conditional density of any future industrial economic state is a normal distribution with the mean and variance as follows:

\[E_t[\eta(s)] = \eta(t) \cdot e^{-a(s-t)} + b_\eta \cdot (1 - e^{-a(s-t)}) \] \hspace{1cm} (A5-2)

\[Var_t[\eta(s)] = \frac{\sigma_\eta^2 [1 - e^{-2a(s-t)}]}{2a} \] \hspace{1cm} (A5-3)

In equation (A5-2) and (A5-3), \(s\) stands for the observed time point in the future.

With this result, we can write the equation as a discrete autoregressive process for order 1, i.e., AR(1) process:

\[\eta(s) = \eta(t) \cdot e^{-a(s-t)} + b_\eta \cdot (1 - e^{-a(s-t)}) + \xi(s)\] \hspace{1cm} (A5-4)

\[\eta_{s+\Delta t} = \eta_s \cdot e^{-a\Delta t} + b_\eta \cdot (1 - e^{-a\Delta t}) + \xi_{s+\Delta t}\] \hspace{1cm} (A5-5)

Where the error term \(\varepsilon\) is normal distributed with mean 0 and variance as described in equation (A5-3). And \(\Delta t\) is a length of time interval. The AR(1) process allows \(\eta_t\) to satisfy all three properties of the OU process, i.e., mean, variance, and white noise with normal density. Obtaining this exact form from discretization is essential for simplifying the estimation process of the parameters. Equation (A5-5) can be written as the following regression model:
\[ \eta_t = \alpha + \beta \cdot \eta_{t-\Delta t} + e_t \quad (A5-6) \]

where \( \alpha = b_\eta (1 - \beta) \), \( \beta = e^{-a\Delta t} \), so all the three parameters can be solved from equation (A5-6).

\[
a = \frac{-\ln \beta}{\Delta t} \quad b_\eta = \frac{\alpha}{1 - \beta_1} \quad \sigma^2_\eta = \frac{2a \cdot MSE}{1 - e^{-2a\Delta t}} \quad (A5-7)
\]

According to equation (A5-7), we therefore estimate the three parameters; that are \( a_\eta \), \( b_\eta \) and \( \sigma_\eta \), of the stochastic industrial economic state model.
Appendix VI. Sensitivity analysis of credit risks’ factors

There exists a firm’s asset value distribution with mean 307 billions and the standard deviation 123 billions. Now we assume that the speed of mean-reversion is equal to one and implement sensitivity analysis for PD and RR in the three situations: the mean of assets value, the standard deviation of assets value, and the debt value.

Figure A6-1. Sensitivity analysis — mean of asset value

Figure A6-2. Sensitivity analysis — standard deviation of asset value
Figure A6-3 shows that PD will decrease and RR will increase when the firm’s value is higher. Figure A6-2 shows that PD will increase and RR will decrease when the firm’s volatility is higher. Figure A6-3 shows that RR is more influenced by debts’ fluctuation than PD. This is because RR is the recovery per unit debt when defaults occur. In summary, the above three figures appear that PD and RR are inverse-related.

Therefore both the mean and the volatility of firm’s value are more sensitive to PD when defaults occur and disposable asset values don’t decrease greatly at the same time. Moreover, the volatility of RR will be larger when the asset values’ fluctuation is high.
Appendix VII. Discussion — seniority and collateralization of corporate debt

1. Assumption: Corporate’ total debt unchanged. Moreover, it equals to provide quite protection on senior debt if raise subordinated debt. In the future, we will construct stochastic interest rate model to adjust the future debts’ changes.

2. Default Decision Rule: \( PV-S < L \)

   - External collaterals
     - \( PV-A(t)-S < L \)
     - Although insurance premium will reduce the firm’s value, but its protection effect make RR easier be estimated.
   - Internal collaterals
     - \( PV-S < L \)
     - Once default occurs, internal collaterals’ value will vary greatly so that RR is hard to be estimated.

3. Multi-period Default Possibility, Recovery rate

Figure A7-1. Flow Chart - CF Based Multi-period Credit Risk Model Decision Rule